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Gauge-invariant description of some (2+1)-dimensional integrable nonlinear evolution equations

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Abstract
New manifestly gauge-invariant forms of two-dimensional generalized dispersive long-wave and Nizhnik–Veselov–Novikov systems of integrable nonlinear equations are presented. It is shown how in different gauges from such forms famous two-dimensional generalization of dispersive long-wave system of equations, Nizhnik–Veselov–Novikov and modified Nizhnik–Veselov–Novikov equations and other known and new integrable nonlinear equations arise. Miura-type transformations between nonlinear equations in different gauges are considered.

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1. Introduction

The fundamental ideas of gauge invariance and gauge transformations are wide spread and in common use in almost every part of physics. The first applications of such ideas in the theory of integrable nonlinear equations by Zakharov and Shabat [1], Kuznetsov and Mikhailov [2], Zakharov and Mikhailov [3], Zakharov and Takhtadzhyan [4], Konopelchenko [5], Konopelchenko and Dubrovsky [6, 7] and others have been made, see also the books [8–13] and references therein.

Now a lot of gauge-equivalent to each other, integrable nonlinear models are well known. In the one-dimensional case the most famous are the nonlinear Schrödinger and Heisenberg ferromagnet equations, massive Thirring model and two-dimensional relativistic field model, KdV and mKdV equations and so on; in the two-dimensional case the most famous are Kadomtsev–Petviashvili and modified Kadomtsev–Petviashvili nonlinear equations, Davey–Stewartson and Ishimori integrable systems of nonlinear equations and so on. See some references in the books [8–14].

In the present paper, manifestly gauge-invariant formulation of two-dimensional nonlinear evolution equations integrable by the following two scalar auxiliary linear problems:

$$L_1\psi = \left(\partial^2_{\xi\eta} + u_1\partial_\xi + v_1\partial_\eta + u_0\right)\psi = 0, \tag{1.1}$$

$$L_2\psi = \left(\partial_t + u_3\partial^3_\xi + v_3\partial^3_\eta + u_2\partial^2_\xi + v_2\partial^2_\eta + \tilde{u}_1\partial_\xi + \tilde{v}_1\partial_\eta + v_0\right)\psi = 0 \tag{1.2}$$

is developed. Here as usual $\xi = x + \sigma y, \eta = x - \sigma y, \sigma^2 = \pm 1$ and $\partial_\xi = \partial/\partial\xi, \partial_\eta = \partial/\partial\eta, \partial^2_\xi = \partial^2/\partial\xi^2$, etc.

Two cases of auxiliary linear problems (1.1), (1.2) with different second auxiliary linear problem (1.2) are studied:

- (i) $u_3 = \kappa_1 = \text{const}, v_3 = \kappa_2 = \text{const}$, third-order problem $L_2\psi = 0$, such choice of second auxiliary problem (1.2) leads to famous Nizhnik–Veselov–Novikov (NVN) [15, 16], modified Nizhnik–Veselov–Novikov (mNVN) [17] and other equations;
- (ii) $u_3 = v_3 = 0, u_2 = \kappa_1 = \text{const}, v_2 = \kappa_2 = \text{const}$, second-order problem $L_2\psi = 0$, such choice of second auxiliary problem (1.2) leads to famous two-dimensional generalization of dispersive long-wave equation (2DDLW) [18], Davey–Stewartson (DS) system of equations [19] and its reductions and other equations.

All above-mentioned famous integrable nonlinear equations via the compatibility condition of auxiliary linear problems (1.1) and (1.2) in the form of Manakov's triad representation [20]

$$[L_1, L_2] = BL_1 \tag{1.3}$$

have been previously established [15–18], see also books [12, 13] and references therein.

In the paper, gauge transformations

$$\psi \to \psi' = g^{-1}\psi \tag{1.4}$$

with arbitrary gauge function $g(\xi, \eta, t)$ of auxiliary linear problems (1.1) and (1.2) are studied. The convenient for gauge-invariant formulation field variables, classical gauge invariants $w_2, \tilde{w}_2, w_1$,

$$w_2 \stackrel{\text{def}}{=} u_0 - u_{1\xi} - u_1v_1 = u'_0 - u'_{1\xi} - u'_1v'_1, \tag{1.5}$$

$$\tilde{w}_2 \stackrel{\text{def}}{=} u_0 - v_{1\eta} - u_1v_1 = u'_0 - v'_{1\eta} - u'_1v'_1, \tag{1.6}$$

$$w_1 \stackrel{\text{def}}{=} u_{1\xi} - v_{1\eta} = u'_{1\xi} - v'_{1\eta} \tag{1.7}$$

and pure gauge variable ρ connected with field variable $u_1(\xi, \eta, t)$ by the formula

$$u_1 \stackrel{\text{def}}{=} (\ln\rho)_\eta \tag{1.8}$$

are introduced. The variable ρ corresponds to pure gauge degrees of freedom and has under (1.4) the following simple law of transformation:

$$\rho \to \rho' = g\rho. \tag{1.9}$$

Let us mention that for the first auxiliary linear problem (1.1), considered as classical partial differential equation, the invariants w_2 and $\tilde{w}_2$ from the early times (see for example the classical book of Forsyth [21]) as Laplace invariants $h = w_2$ and $k = \tilde{w}_2$ are known.

The main results of the paper are the following new integrable systems of nonlinear equations in terms of field variables ρ, w_1, w_2 given by (1.5)–(1.8).

In the case (i) of third-order linear auxiliary problem (1.2) the first invariant w_1 is equal to zero $w_1 \equiv 0$ and the established integrable system of nonlinear equations in terms of ρ, w_2 has the form

$$\rho_t = -\kappa_1 \rho_{\xi\xi\xi} - \kappa_2 \rho_{\eta\eta\eta} - 3\kappa_1 \rho_\xi \partial_\eta^{-1} w_{2\xi} - 3\kappa_2 \rho_\eta \partial_\xi^{-1} w_{2\eta} + v_0 \rho, \tag{1.10}$$

$$w_{2t} = -\kappa_1 w_{2\xi\xi\xi} - \kappa_2 w_{2\eta\eta\eta} - 3\kappa_1 \left(w_2 \partial_\eta^{-1} w_{2\xi}\right)_\xi - 3\kappa_2 \left(w_2 \partial_\xi^{-1} w_{2\eta}\right)_\eta. \tag{1.11}$$

It is remarkable that the gauge-invariant subsystem of the system (1.10)–(1.11), equation (1.11) for the gauge invariant $w_2 = u_0 - u_{1\xi} - u_1 v_1$, coincides in form with the famous NVN equation [15, 16]

$$u_t = -\kappa_1 u_{\xi\xi\xi} - \kappa_2 u_{\eta\eta\eta} - 3\kappa_1 \left(u \partial_\eta^{-1} u_\xi\right)_\xi - 3\kappa_2 \left(u \partial_\xi^{-1} u_\eta\right)_\eta. \tag{1.12}$$

Due to the last remark the system (1.10)–(1.11) will be named below as the Nizhnik–Veselov–Novikov (NVN) system of equations.

In the case (ii) of second-order linear auxiliary problem (1.2) the established integrable system of nonlinear equations in terms of ρ, w_1 and w_2 has the form

$$\rho_t = -\kappa_1 \rho_{\xi\xi} - \kappa_2 \rho_{\eta\eta} - 2\kappa_1 \rho \partial_\eta^{-1} w_{2\xi} + 2\kappa_2 \rho_\eta \partial_\xi^{-1} w_1 + v_0 \rho, \tag{1.13}$$

$$w_{1t} = -\kappa_1 w_{1\xi\xi} + \kappa_2 w_{1\eta\eta} - 2\kappa_1 w_{2\xi\xi} + 2\kappa_2 w_{2\eta\eta} - 2\kappa_1 \left(w_1 \partial_\eta^{-1} w_1\right)_\xi + 2\kappa_2 \left(w_1 \partial_\xi^{-1} w_1\right)_\eta, \tag{1.14}$$

$$w_{2t} = \kappa_1 w_{2\xi\xi} - \kappa_2 w_{2\eta\eta} - 2\kappa_1 \left(w_2 \partial_\eta^{-1} w_1\right)_\xi + 2\kappa_2 \left(w_2 \partial_\xi^{-1} w_1\right)_\eta. \tag{1.15}$$

The gauge-invariant subsystem of the system (1.13)–(1.15), the system of equations (1.14)–(1.15) for invariants $w_1 = u_{1\xi} - v_{1\eta}$ and $w_2 = u_0 - u_{1\xi} - u_1 v_1$, for the choice $u_1 = 0$, $v_1 = v$, $u_0 = u$ for which $w_1 = -v_\eta$, $w_2 = u$, leads to the well-known system of equations [22]

$$v_t = -\kappa_1 v_{\xi\xi} + \kappa_2 v_{\eta\eta} + 2\kappa_1 \partial_\eta^{-1} u_{\xi\xi} - 2\kappa_2 u_\eta + 2\kappa_1 v v_\xi - 2\kappa_2 v_\eta \partial_\xi^{-1} v_\eta, \tag{1.16}$$

$$u_t = \kappa_1 u_{\xi\xi} - \kappa_2 u_{\eta\eta} + 2\kappa_1 (uv)_\xi - 2\kappa_2 \left(u \partial_\xi^{-1} v_\eta\right)_\eta. \tag{1.17}$$

In terms of variables

$$v = -\frac{q}{2}, \qquad u = \frac{1}{4}(1 + r - q_\eta) \tag{1.18}$$

the integrable system of nonlinear equations (1.16)–(1.17) takes the form

$$q_t = -\kappa_1 \partial_\eta^{-1} r_{\xi\xi} + \kappa_2 r_\eta - \frac{\kappa_1}{2}(q^2)_\xi + \kappa_2 q_\eta \partial_\xi^{-1} q_\eta, \tag{1.19}$$

$$r_t = -\kappa_1 q_\xi + \kappa_2 \partial_\xi^{-1} q_{\eta\eta} - \kappa_1 q_{\eta\xi\xi} + \kappa_2 q_{\eta\eta\eta} - \kappa_1 (rq)_\xi + \kappa_2 \left(r \partial_\xi^{-1} q_\eta\right)_\eta. \tag{1.20}$$

For the particular value $\kappa_2 = 0$ the system of equations (1.19)–(1.20) reduces to the famous integrable two-dimensional generalization of dispersive long-wave system of equations [18]

$$q_{t\eta} = -\kappa_1 r_{\xi\xi} - \frac{\kappa_1}{2}(q^2)_{\xi\eta}, \tag{1.21}$$

$$r_{t\xi} = -\kappa_1 (qr + q + q_{\xi\eta})_{\xi\xi}. \tag{1.22}$$

In one-dimensional limit $\xi = \eta$ both systems (1.19)–(1.20) with $\kappa_1 - \kappa_2 = 1$ and (1.21)–(1.22) with $\kappa_1 = 1$ reduce to the famous dispersive long-wave equation (see, e.g., Broer [23]). It is worthwhile by this reason to name the system (1.13)–(1.15) as the two-dimensional generalized dispersive long-wave (2DGDLW) system of equations.

In both considered cases of the third- and second-order auxiliary linear problem (1.2) the integrable systems of nonlinear equations (1.10)–(1.11) and (1.13)–(1.15) have common gauge-transparent structure. They contain correspondingly:

- gauge-invariant subsystems (1.11) and (1.14)–(1.15);
- equations (1.10) and (1.13) for the pure gauge variable ρ with some terms containing gauge invariants.

For the zero values of invariants $w_1 = 0$, $w_2 = 0$ both systems (1.10)–(1.11) and (1.13)–(1.15) reduce to corresponding linear equations for ρ, respectively,

$$\rho_t = -\kappa_1 \rho_{\xi\xi\xi} - \kappa_2 \rho_{\eta\eta\eta} + v_0 \rho \tag{1.23}$$

and

$$\rho_t = -\kappa_1 \rho_{\xi\xi} - \kappa_2 \rho_{\eta\eta} + v_0 \rho. \tag{1.24}$$

In this paper the NVN (1.10)–(1.11) and the 2DGDLW (1.13)–(1.15) systems of integrable nonlinear equations in different gauges are considered.

It is shown that in some different gauges from (1.10)–(1.11) famous Nizhnik–Veselov–Novikov (NVN) [15, 16] and modified Nizhnik–Veselov–Novikov (mNVN) [17] equations follow, these equations by Miura-type transformation are connected.

It is shown also that the gauge-invariant subsystem (1.14)–(1.15) of the 2DGDLW system (1.13)–(1.15) contains in particular, the famous case, integrable two-dimensional generalization of dispersive long-wave system [18] of integrable nonlinear equations. In some cases the special gauge 2DGDLW system (1.13)–(1.15) reduces to the famous Davey–Stewartson (DS) system [19] of nonlinear equations and in another special gauges to new DS-type systems of integrable nonlinear equations, these systems by Miura-type transformation are connected.

The plan of our paper is the following. In sections 2 and 3 via the compatibility condition (1.3) the manifestly gauge-invariant correspondingly integrable NVN system (1.10)–(1.11) and the 2DGDLW system (1.13)–(1.15) of nonlinear equations are derived. Some special gauges of NVN (1.10)–(1.11) and 2DGDLW (1.13)–(1.15) integrable systems of nonlinear equations are considered. Miura-type transformations between solutions of nonlinear equations in different gauges are established.

2. Manifestly gauge-invariant formulation of NVN system of equations

It is instructive to derive integrable nonlinear equations starting from auxiliary linear problems (1.1) and (1.2) in general position, with all nonzero field variables.

Using the compatibility condition (1.3) in the form of Manakov's triad representation [20] after some calculations one obtains, equating to zero the coefficients at different degrees of partial derivatives $\partial_\xi^n \partial_\eta^m$ of the relation $[L_1, L_2] - BL_1 = 0$, the following system of equations for the field variables $u_3, v_3, u_2, v_2, \tilde{u}_1, \tilde{v}_1, v_0$ and u_1, v_1, u_0:

$$\partial_\xi^4 : u_{3\eta} = 0, \qquad \partial_\eta^4 : v_{3\xi} = 0, \tag{2.1}$$

$$\partial_\xi^3 \partial_\eta : u_{3\xi} = 0, \qquad \partial_\xi \partial_\eta^3 : v_{3\eta} = 0, \tag{2.2}$$

$$\partial_\xi^3 : u_{3\xi\eta} + u_{2\eta} + u_1 u_{3\xi} - 3u_3 u_{1\xi} + v_1 u_{3\eta} = 0, \tag{2.3}$$

$$\partial_\eta^3 : v_{3\xi\eta} + v_{2\xi} + v_1 v_{3\eta} - 3v_3 v_{1\eta} + u_1 v_{3\xi} = 0, \tag{2.4}$$

$$\partial_\xi^2\partial_\eta : u_{2\xi} - 3u_3 v_{1\xi} = 0, \qquad \partial_\xi\partial_\eta^2 : v_{2\eta} - 3v_3 u_{1\eta} = 0, \tag{2.5}$$

$$\partial_\xi^2 : u_{2\xi\eta} + \tilde{u}_{1\eta} - 3u_3 u_{1\xi\xi} - 2u_2 u_{1\xi} + u_1 u_{2\xi} + v_1 u_{2\eta} - 3u_3 u_{0\xi} = 0, \tag{2.6}$$

$$\partial_\eta^2 : v_{2\xi\eta} + \tilde{v}_{1\xi} - 3v_3 v_{1\eta\eta} - 2v_2 v_{1\eta} + u_1 v_{2\xi} + v_1 v_{2\eta} - 3v_3 u_{0\eta} = 0, \tag{2.7}$$

$$\partial_{\xi\eta}^2 : \tilde{u}_{1\xi} + \tilde{v}_{1\eta} - 3u_3 v_{1\xi\xi} - 3v_3 u_{1\eta\eta} - 2u_2 v_{1\xi} - 2v_2 u_{1\eta} - B = 0, \tag{2.8}$$

$$\begin{aligned} -\partial_\xi : u_{1t} &+ u_3 u_{1\xi\xi\xi} + v_3 u_{1\eta\eta\eta} + u_2 u_{1\xi\xi} + v_2 u_{1\eta\eta} - v_{0\eta} + \tilde{u}_1 u_{1\xi} + \tilde{v}_1 u_{1\eta} \\ &- u_1 \tilde{u}_{1\xi} - v_1 \tilde{u}_{1\eta} - \tilde{u}_{1\xi\eta} + 3u_3 u_{0\xi\xi} + 2u_2 u_{0\xi} + B u_1 = 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} -\partial_\eta : v_{1t} &+ u_3 v_{1\xi\xi\xi} + v_3 v_{1\eta\eta\eta} + u_2 v_{1\xi\xi} + v_2 v_{1\eta\eta} - v_{0\xi} + \tilde{v}_1 v_{1\eta} + \tilde{u}_1 v_{1\xi} \\ &- u_1 \tilde{v}_{1\xi} - v_1 \tilde{v}_{1\eta} - \tilde{v}_{1\xi\eta} + 3v_3 u_{0\eta\eta} + 2v_2 u_{0\eta} + B v_1 = 0, \end{aligned} \tag{2.10}$$

$$\begin{aligned} -\partial^0 : u_{0t} &+ u_3 u_{0\xi\xi\xi} + v_3 u_{0\eta\eta\eta} + u_2 u_{0\xi\xi} + v_2 u_{0\eta\eta} + \tilde{u}_1 u_{0\xi} + \tilde{v}_1 u_{0\eta} \\ &- u_1 v_{0\xi} - v_1 v_{0\eta} - v_{0\xi\eta} + B u_0 = 0. \end{aligned} \tag{2.11}$$

The system of defining equations (2.1)–(2.11) has recurrent character and allows us to express via (2.1)–(2.7) the field variables u_3, v_3, u_2, v_2 and $\tilde{u}_1, \tilde{v}_1$ of the second auxiliary problem (1.2) through the field variables u_1, v_1, u_0 of the first auxiliary linear problem (1.1). The last three equations (2.9)–(2.11) represent the integrable system of nonlinear evolution equations for the field variables u_1, v_1 and u_0.

In the case of the second auxiliary linear problem (1.2) of third order from relations (2.1) and (2.2) it follows that the coefficients u_3 and v_3 are constants,

$$u_3 = \text{const} = \kappa_1, \qquad v_3 = \text{const} = \kappa_2. \tag{2.12}$$

Using (2.12) one obtains from the relations (2.3)–(2.5),

$$u_{2\xi} = 3\kappa_1 v_{1\xi}, \qquad v_{2\eta} = 3\kappa_2 u_{1\eta}, \tag{2.13}$$

$$u_{2\eta} = 3\kappa_1 u_{1\xi}, \qquad v_{2\xi} = 3\kappa_2 v_{1\eta}. \tag{2.14}$$

From (2.13)–(2.14) the important relation between field variables u_1, v_1,

$$u_{1\xi} = v_{1\eta} \tag{2.15}$$

and expressions for variables u_2 and v_2,

$$u_2 = 3\kappa_1 v_1 + \text{const}_1, \qquad v_2 = 3\kappa_2 u_1 + \text{const}_2, \tag{2.16}$$

follow. Arising in (2.16), for simplicity the constants being equal to zero are chosen below. By the use of (2.6) and (2.7) taking into account (2.12), (2.15) and (2.16) one derives the expressions for $\tilde{u}_1$ and $\tilde{v}_1$,

$$\tilde{u}_1 = 3\kappa_1 \partial_\eta^{-1} u_{0\xi} - 3\kappa_1 \partial_\eta^{-1}(u_1 v_{1\xi}) + \frac{3\kappa_1}{2} v_1^2 + f_1(\xi, t), \tag{2.17}$$

$$\tilde{v}_1 = 3\kappa_2 \partial_\xi^{-1} u_{0\eta} - 3\kappa_2 \partial_\xi^{-1}(v_1 u_{1\eta}) + \frac{3\kappa_2}{2} u_1^2 + g_1(\eta, t), \tag{2.18}$$

including as 'constants' of integration the arbitrary functions $f_1(\xi, t)$ and $g_1(\eta, t)$ which for simplicity are chosen below as equal to zero values. Inserting $\tilde{u}_1$ and $\tilde{v}_1$ from (2.17), (2.18) into (2.8) and taking into account (1.5), (2.12), (2.15)–(2.18) one derives the expression for the coefficient B,

$$\begin{aligned} B = &-3\kappa_1 v_{1\xi\xi} - 3\kappa_2 u_{1\eta\eta} - 3\kappa_1 v_1 v_{1\xi} - 3\kappa_2 u_1 u_{1\eta} + 3\kappa_1 \partial_\eta^{-1} u_{0\xi\xi} + 3\kappa_2 \partial_\xi^{-1} u_{0\eta\eta} \\ &- 3\kappa_1 \partial_\eta^{-1}(u_1 v_{1\xi})_\xi - 3\kappa_2 \partial_\xi^{-1}(v_1 u_{1\eta})_\eta = 3\kappa_1 \partial_\eta^{-1} w_{2\xi\xi} + 3\kappa_2 \partial_\xi^{-1} w_{2\eta\eta}. \end{aligned} \tag{2.19}$$

The last three equations (2.9)–(2.11) of the system (2.1)–(2.11) are the evolution equations for the field variables u_1, v_1 and u_0. By the use of (1.5), (2.12), (2.15)–(2.19) after some calculations (by singling out in some terms the combination of field variables

$w_2 = u_0 - u_{1\xi} - u_1 v_1$ coinciding with gauge invariant (1.5)) these equations can be represented in the following convenient form:

$$u_{1t} = -\kappa_1 u_{1\xi\xi\xi} - \kappa_2 u_{1\eta\eta\eta} - \kappa_1\left(v_1^3 + 3v_1 v_{1\xi}\right)_\eta - \kappa_2\left(u_1^3 + 3u_1 u_{1\eta}\right)_\eta - 3\kappa_1\left(v_1 \partial_\eta^{-1} w_{2\xi}\right)_\eta - 3\kappa_2\left(u_1 \partial_\xi^{-1} w_{2\eta}\right)_\eta + v_{0\eta}, \tag{2.20}$$

$$v_{1t} = -\kappa_1 v_{1\xi\xi\xi} - \kappa_2 v_{1\eta\eta\eta} - \kappa_1\left(v_1^3 + 3v_1 v_{1\xi}\right)_\xi - \kappa_2\left(u_1^3 + 3u_1 u_{1\eta}\right)_\xi - 3\kappa_1\left(v_1 \partial_\eta^{-1} w_{2\xi}\right)_\xi - 3\kappa_2\left(u_1 \partial_\xi^{-1} w_{2\eta}\right)_\xi + v_{0\xi}, \tag{2.21}$$

$$u_{0t} = -\kappa_1 u_{0\xi\xi\xi} - \kappa_2 u_{0\eta\eta\eta} - 3\kappa_1 v_1 u_{0\xi\xi} - 3\kappa_2 u_1 u_{0\eta\eta} - 3\kappa_1\left(v_{1\xi} + v_1^2\right)u_{0\xi} - 3\kappa_2\left(u_{1\eta} + u_1^2\right)u_{0\eta} - 3\kappa_1\left(u_0 \partial_\eta^{-1} w_{2\xi}\right)_\xi - 3\kappa_2\left(u_0 \partial_\xi^{-1} w_{2\eta}\right)_\eta + v_{0\xi\eta} + u_1 v_{0\xi} + v_1 v_{0\eta}. \tag{2.22}$$

Remember that in the considered case due to (2.15) the first invariant $w_1 = u_{1\xi} - v_{1\eta} = 0$ is equal to zero.

Due to the equality $u_{1\xi} = v_{1\eta}$ one can reduce the set of dependent variables u_1, v_1 and u_0 in the system (2.20)–(2.22) to two variables ρ, w_2 (or equivalently to variables $\phi = \ln\rho$, w_2) defined by the relations

$$u_1 \stackrel{\text{def}}{=} \phi_\eta = \frac{\rho_\eta}{\rho}, \qquad v_1 \stackrel{\text{def}}{=} \phi_\xi = \frac{\rho_\xi}{\rho}, \tag{2.23}$$

$$w_2 = u_0 - u_{1\xi} - u_1 v_1 = u_0 - \phi_{\xi\eta} - \phi_\xi \phi_\eta = u_0 - \frac{\rho_{\xi\eta}}{\rho}. \tag{2.24}$$

Indeed the insertion of $u_1 = \phi_\eta$ and $v_1 = \phi_\xi$ into (2.20) and (2.21) reduces both these equations to the single one equation

$$\phi_t = -\kappa_1 \phi_{\xi\xi\xi} - \kappa_2 \phi_{\eta\eta\eta} - \kappa_1 (\phi_\xi)^3 - \kappa_2 (\phi_\eta)^3 - 3\kappa_1 \phi_\xi \phi_{\xi\xi} - 3\kappa_2 \phi_\eta \phi_{\eta\eta} - 3\kappa_1 \phi_\xi \partial_\eta^{-1} w_{2\xi} - 3\kappa_2 \phi_\eta \partial_\xi^{-1} w_{2\eta} + v_0, \tag{2.25}$$

or in terms of variables ρ, w_2 to the equation

$$\rho_t = -\kappa_1 \rho_{\xi\xi\xi} - \kappa_2 \rho_{\eta\eta\eta} - 3\kappa_1 \rho_\xi \partial_\eta^{-1} w_{2\xi} - 3\kappa_2 \rho_\eta \partial_\xi^{-1} w_{2\eta} + v_0 \rho. \tag{2.26}$$

One can show also that the exclusion of field variable v_0 from the last equation (2.22) by the use of derivatives $v_{0\xi}$, $v_{0\eta}$ and $v_{0\xi\eta}$ (calculated from the first two equations (2.20) and (2.21)) leads to the following nonlinear evolution equation for the second invariant w_2:

$$w_{2t} = -\kappa_1 w_{2\xi\xi\xi} - \kappa_2 w_{2\eta\eta\eta} - 3\kappa_1\left(w_2 \partial_\eta^{-1} w_{2\xi}\right)_\xi - 3\kappa_2\left(w_2 \partial_\xi^{-1} w_{2\eta}\right)_\eta. \tag{2.27}$$

So by the change of variables (2.23), (2.24) the integrable system of nonlinear equations (2.20)–(2.22) is reduced to the following equivalent integrable system of nonlinear equations:

$$\rho_t = -\kappa_1 \rho_{\xi\xi\xi} - \kappa_2 \rho_{\eta\eta\eta} - 3\kappa_1 \rho_\xi \partial_\eta^{-1} w_{2\xi} - 3\kappa_2 \rho_\eta \partial_\xi^{-1} w_{2\eta} + v_0 \rho, \tag{2.28}$$

$$w_{2t} = -\kappa_1 w_{2\xi\xi\xi} - \kappa_2 w_{2\eta\eta\eta} - 3\kappa_1\left(w_2 \partial_\eta^{-1} w_{2\xi}\right)_\xi - 3\kappa_2\left(w_2 \partial_\xi^{-1} w_{2\eta}\right)_\eta. \tag{2.29}$$

Equivalently, in terms of variables $\phi = \ln\rho$ and w_2, the system of equations (2.28)–(2.29) takes the form

$$\phi_t = -\kappa_1 \phi_{\xi\xi\xi} - \kappa_2 \phi_{\eta\eta\eta} - \kappa_1 (\phi_\xi)^3 - \kappa_2 (\phi_\eta)^3 - 3\kappa_1 \phi_\xi \phi_{\xi\xi} - 3\kappa_2 \phi_\eta \phi_{\eta\eta} - 3\kappa_1 \phi_\xi \partial_\eta^{-1} w_{2\xi} - 3\kappa_2 \phi_\eta \partial_\xi^{-1} w_{2\eta} + v_0, \tag{2.30}$$

$$w_{2t} = -\kappa_1 w_{2\xi\xi\xi} - \kappa_2 w_{2\eta\eta\eta} - 3\kappa_1\left(w_2 \partial_\eta^{-1} w_{2\xi}\right)_\xi - 3\kappa_2\left(w_2 \partial_\xi^{-1} w_{2\eta}\right)_\eta. \tag{2.31}$$

Note that equation (2.29) (or (2.31)) for the gauge invariant w_2 exactly coincides in form with the famous NVN equation [15, 16]. Due to this reason it is worthwhile to name the integrable systems (2.28)–(2.29) (or (2.30)–(2.31)) as the NVN system of equations.

The NVN system of equations (2.28)–(2.29) (or (2.30)–(2.31)) has gauge-transparent structure. It contains:

- explicitly gauge-invariant subsystem—equation (2.29) (or (2.31)) for invariant w_2;
- equation (2.28) (or (2.30)) for pure gauge variable ρ (or ϕ) with some terms containing gauge invariant w_2 and field variable v_0 from the second linear auxiliary problem (1.2).

Manakov's triad representation (1.3) for the NVN system of equations (2.28)–(2.29) (or (2.30)–(2.31)), due to formulae (2.12)–(2.19) and (2.23)–(2.24), includes the following operators L_1, L_2 of auxiliary linear problems and coefficient $B(w_2)$:

$$L_1 = \partial^2_{\xi\eta} + \frac{\rho_\eta}{\rho}\partial_\xi + \frac{\rho_\xi}{\rho}\partial_\eta + w_2 + \frac{\rho_{\xi\eta}}{\rho}, \tag{2.32}$$

$$\begin{aligned} L_2 = \partial_t + \kappa_1\partial^3_\xi + \kappa_2\partial^3_\eta + 3\kappa_1\frac{\rho_\xi}{\rho}\partial^2_\xi + 3\kappa_2\frac{\rho_\eta}{\rho}\partial^2_\eta + 3\kappa_1\left(\frac{\rho_{\xi\xi}}{\rho} + \left(\partial_\eta^{-1}w_{2\xi}\right)\right)\partial_\xi \\ + 3\kappa_2\left(\frac{\rho_{\eta\eta}}{\rho} + \left(\partial_\xi^{-1}w_{2\eta}\right)\right)\partial_\eta + v_0, \end{aligned} \tag{2.33}$$

$$B(w_2) = 3\kappa_1\partial_\eta^{-1}w_{2\xi\xi} + 3\kappa_2\partial_\xi^{-1}w_{2\eta\eta}. \tag{2.34}$$

In the case $w_2 = 0$ of zero invariant the NVN system of equations (2.28)–(2.29) (or (2.30)–(2.31)) reduces to linear equation

$$\rho_t = -\kappa_1\rho_{\xi\xi\xi} - \kappa_2\rho_{\eta\eta\eta} + v_0\rho, \tag{2.35}$$

which is integrable by auxiliary linear problems (1.1) and (1.2) with L_1 and L_2 from (2.32), (2.33) under $w_2 = 0$. The compatibility condition in this case, due to $B(w_2) = 0$, has Lax form $[L_1, L_2] = 0$. In terms of variable $\phi = \ln\rho$ linear equation (2.35) looks like Burgers-type equation of the third order

$$\phi_t = -\kappa_1\phi_{\xi\xi\xi} - \kappa_2\phi_{\eta\eta\eta} - \kappa_1(\phi_\xi)^3 - \kappa_2(\phi_\eta)^3 - 3\kappa_1\phi_\xi\phi_{\xi\xi} - 3\kappa_2\phi_\eta\phi_{\eta\eta} + v_0, \tag{2.36}$$

which linearizes by the substitution $\phi = \ln\rho$ and consequently is C-integrable.

Let us denote by $C(\phi, u_0, v_0)$ the gauge which corresponds to nonzero field variables $u_1 = \phi_\eta$, $v_1 = \phi_\xi$, u_0 and v_0 of linear problems (1.1) and (1.2) and consequently to NVN system (2.30)–(2.31) in general position. Under different gauges from NVN system different integrable nonlinear equations follow, which are gauge-equivalent to each other. The solutions of these equations by some Miura-type transformation are connected.

For example in the gauge $C(0, u_0, 0)$ the NVN system of equations (2.30)–(2.31) reduces to the famous NVN equation [15, 16] for the field variable u_0,

$$u_{0t} = -\kappa_1 u_{0\xi\xi\xi} - \kappa_2 u_{0\eta\eta\eta} - 3\kappa_1\left(u_0\partial_\eta^{-1}u_{0\xi}\right)_\xi - 3\kappa_2\left(u_0\partial_\xi^{-1}u_{0\eta}\right)_\eta. \tag{2.37}$$

In another gauge $C(\phi, 0, v_0)$ the NVN system (2.30)–(2.31) takes the form

$$\phi_t = -\kappa_1\phi_{\xi\xi\xi} - \kappa_2\phi_{\eta\eta\eta} - \kappa_1(\phi_\xi)^3 - \kappa_2(\phi_\eta)^3 + 3\kappa_1\phi_\xi\partial_\eta^{-1}(\phi_\xi\phi_\eta)_\xi + 3\kappa_2\phi_\eta\partial_\xi^{-1}(\phi_\xi\phi_\eta)_\eta + v_0, \tag{2.38}$$

$$\begin{aligned} \left(\partial^2_{\xi\eta} + \phi_\eta\partial_\xi + \phi_\xi\partial_\eta\right)\phi_t = \left(\partial^2_{\xi\eta} + \phi_\eta\partial_\xi + \phi_\xi\partial_\eta\right)\big[-\kappa_1\phi_{\xi\xi\xi} - \kappa_2\phi_{\eta\eta\eta} - \kappa_1(\phi_\xi)^3 \\ - \kappa_2(\phi_\eta)^3 + 3\kappa_1\phi_\xi\partial_\eta^{-1}(\phi_\xi\phi_\eta)_\xi + 3\kappa_2\phi_\eta\partial_\xi^{-1}(\phi_\xi\phi_\eta)_\eta\big], \end{aligned} \tag{2.39}$$

and consequently to the following system of equations:

$$\phi_t = -\kappa_1\phi_{\xi\xi\xi} - \kappa_2\phi_{\eta\eta\eta} - \kappa_1(\phi_\xi)^3 - \kappa_2(\phi_\eta)^3 + 3\kappa_1\phi_\xi\partial_\eta^{-1}(\phi_\xi\phi_\eta)_\xi + 3\kappa_2\phi_\eta\partial_\xi^{-1}(\phi_\xi\phi_\eta)_\eta + v_0, \tag{2.40}$$

$$\left(\partial_{\xi\eta}^2 + \phi_\eta\partial_\xi + \phi_\xi\partial_\eta\right)v_0 = 0 \tag{2.41}$$

is equivalent. For $v_0 = 0$ system of equations (2.40)–(2.41) reduces to the famous modified Nizhnik–Veselov–Novikov equation

$$\phi_t = -\kappa_1\phi_{\xi\xi\xi} - \kappa_2\phi_{\eta\eta\eta} - \kappa_1(\phi_\xi)^3 - \kappa_2(\phi_\eta)^3 + 3\kappa_1\phi_\xi\partial_\eta^{-1}(\phi_\xi\phi_\eta)_\xi + 3\kappa_2\phi_\eta\partial_\xi^{-1}(\phi_\xi\phi_\eta)_\eta, \tag{2.42}$$

which at first in the paper [17] of Konopelchenko in a different context was discovered. Let us mention that the considered version (2.42) of mNVN equation derived in the present paper in the framework of manifestly gauge-invariant description is different from the mNVN equation discovered in the paper [24].

The new system of equations (2.40)–(2.41) can be named as modified NVN (mNVN) system of equations. This system due to (2.32)–(2.34) and to the choice of the gauge $C(\phi, 0, v_0)$ has the following triad representation (1.3) with triad (L_1, L_2, B):

$$L_1 = \partial_{\xi\eta}^2 + \phi_\eta\partial_\xi + \phi_\xi\partial_\eta, \tag{2.43}$$

$$\begin{aligned} L_2 = \partial_t &+ \kappa_1\partial_\xi^3 + \kappa_2\partial_\eta^3 + 3\kappa_1\phi_\xi\partial_\xi^2 + 3\kappa_2\phi_\eta\partial_\eta^2 + 3\kappa_1\left(\phi_\xi^2 - \partial_\eta^{-1}(\phi_\xi\phi_\eta)_\xi\right)\partial_\xi \\ &+ 3\kappa_2\left(\phi_\eta^2 - \partial_\xi^{-1}(\phi_\xi\phi_\eta)_\eta\right)\partial_\eta + v_0, \end{aligned} \tag{2.44}$$

$$B(w_2) = -3\kappa_1\phi_{\xi\xi\xi} - 3\kappa_2\phi_{\eta\eta\eta} - 3\kappa_1\partial_\eta^{-1}(\phi_\xi\phi_\eta)_{\xi\xi} - 3\kappa_2\partial_\xi^{-1}(\phi_\xi\phi_\eta)_{\eta\eta}. \tag{2.45}$$

The mNVN equation (2.42) has triad representation (2.43)–(2.45) with $v_0 = 0$.

It is evident that the solutions u_0 and ϕ of NVN (2.37) and mNVN (2.42) equations via invariant $w_2 = u_0 = -\phi_{\xi\eta} - \phi_\xi\phi_\eta$ (calculated in different gauges $C(0, u_0, 0)$ and $C(\phi, 0, 0)$) by Miura-type transformation

$$u_0 = -\phi_{\xi\eta} - \phi_\xi\phi_\eta \tag{2.46}$$

are connected. In a one-dimensional limit, under $\partial_\xi = \partial_\eta$, the mNVN equation (2.42) reduces to the mKdV equation in potential form

$$\phi_t = -\kappa\phi_{\xi\xi\xi} + 2\kappa(\phi_\xi)^3, \tag{2.47}$$

where $\kappa = \kappa_1 + \kappa_2$. In terms of variable $v_1 = \phi_\xi$ this is mKdV equation

$$v_{1t} = -\kappa v_{1\xi\xi\xi} + 6\kappa v_1^2 v_{1\xi}. \tag{2.48}$$

3. Manifestly gauge-invariant formulation of two-dimensional generalization of the dispersive long-wave equations system

In the case of the second-order linear auxiliary problem (1.2) the coefficients u_3, v_3 in the system of relations (2.1)–(2.11) have zero values $u_3 = v_3 = 0$. The relations (2.3)–(2.5) lead to constant values for the coefficients u_2 and v_2,

$$u_2 = \text{const} = \kappa_1, \qquad v_2 = \text{const} = \kappa_2. \tag{3.1}$$

By integration of relations (2.6) and (2.7) one immediately obtains the expressions for the coefficients $\tilde{u}_1$ and $\tilde{v}_1$,

$$\tilde{u}_1 = 2\kappa_1 \partial_\eta^{-1} u_{1\xi} + f_2(\xi, t), \qquad \tilde{v}_1 = 2\kappa_2 \partial_\xi^{-1} v_{1\eta} + g_2(\eta, t), \tag{3.2}$$

where $f_2(\xi, t)$ and $g_2(\eta, t)$ are arbitrary functions which below, for simplicity, are chosen equal to zero values. Inserting (3.1)–(3.2) into (2.8) one obtains taking into account (1.7) the expression for coefficient B,

$$B = -2\kappa_1 v_{1\xi} - 2\kappa_2 u_{1\eta} + 2\kappa_1 \partial_\eta^{-1} u_{1\xi\xi} + 2\kappa_2 \partial_\xi^{-1} v_{1\eta\eta} = 2\kappa_1 \partial_\eta^{-1} w_{1\xi} - 2\kappa_2 \partial_\xi^{-1} w_{1\eta}. \tag{3.3}$$

The last three relations (2.9)–(2.11) of the system (2.1)–(2.11) are nonlinear evolution equations for the field variables u_1, v_1 and u_0. By the use of (3.1)–(3.3) after some calculations these equations can be represented (by singling out in some terms the combinations of field variables $w_1 = u_{1\xi} - v_{1\eta}$ and $w_2 = u_0 - u_{1\xi} - u_1 v_1$ coinciding with gauge invariants (1.5)–(1.7)) in the following convenient form:

$$\begin{aligned} u_{1t} = {} & -\kappa_1 v_{1\xi\eta} - \kappa_2 u_{1\eta\eta} - 2\kappa_2 u_1 u_{1\eta} - \kappa_1 w_{1\xi} - 2\kappa_1 w_{2\xi} - 2\kappa_1 u_{1\xi} \partial_\eta^{-1} u_{1\xi} \\ & + 2\kappa_2 \left(u_1 \partial_\xi^{-1} w_1\right)_\eta + v_{0\eta}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} v_{1t} = {} & -\kappa_1 v_{1\xi\xi} - \kappa_2 u_{1\xi\eta} - 2\kappa_1 v_1 v_{1\xi} - \kappa_2 w_{1\eta} - 2\kappa_2 w_{2\eta} - 2\kappa_2 v_{1\eta} \partial_\xi^{-1} v_{1\eta} \\ & - 2\kappa_1 \left(v_1 \partial_\eta^{-1} w_1\right)_\xi + v_{0\xi}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} u_{0t} = {} & -\kappa_1 u_{0\xi\xi} - \kappa_2 u_{0\eta\eta} - 2\kappa_1 u_{0\xi} v_1 - 2\kappa_2 u_{0\eta} u_1 - 2\kappa_1 \left(u_0 \partial_\eta^{-1} w_1\right)_\xi \\ & + 2\kappa_2 \left(u_0 \partial_\xi^{-1} w_1\right)_\eta + v_{0\xi\eta} + u_1 v_{0\xi} + v_1 v_{0\eta}. \end{aligned} \tag{3.6}$$

Let us emphasize that the integrable system of nonlinear equations (3.4)–(3.6) arises as a compatibility condition of auxiliary linear problems (1.1) and (1.2) in the form (1.3) of Manakov's triad representation in the general position. The system contains three evolution equations for the field variables u_1, v_1 and u_0. These equations include also the field variable v_0 from the second auxiliary linear problem. The presence of these four dependent variables u_1, v_1, u_0 and v_0 in system (3.4)–(3.6) of three nonlinear equations reflects gauge freedom of auxiliary linear problems (1.1) and (1.2) and the corresponding integrable systems of nonlinear equations. In contrast to the case considered in the, previous section the first invariant $w_1 = u_{1\xi} - v_{1\eta} \neq 0$ is not equal to zero.

One can show that the first two equations (3.4) and (3.5) of the last system under change of variables

$$u_1 = \phi_\eta = \frac{\rho_\eta}{\rho}, \qquad v_1 = -\partial_\eta^{-1} w_1 + \phi_\xi = -\partial_\eta^{-1} w_1 + \frac{\rho_\xi}{\rho}, \tag{3.7}$$

$$w_2 = u_0 - \phi_{\xi\eta} - \phi_\xi \phi_\eta + \phi_\eta \partial_\eta^{-1} w_1 = u_0 - \frac{\rho_{\xi\eta}}{\rho} + \frac{\rho_\eta}{\rho} \partial_\eta^{-1} w_1, \tag{3.8}$$

reduce to the single equation of the form

$$\rho_t = -\kappa_1 \rho_{\xi\xi} - \kappa_2 \rho_{\eta\eta} - 2\kappa_1 \rho \partial_\eta^{-1} w_{2\xi} + 2\kappa_2 \rho_\eta \partial_\xi^{-1} w_1 + v_0 \rho, \tag{3.9}$$

or in terms of variable $\phi = \ln \rho$ to the equation

$$\phi_t = -\kappa_1 \phi_{\xi\xi} - \kappa_2 \phi_{\eta\eta} - \kappa_1 (\phi_\xi)^2 - \kappa_2 (\phi_\eta)^2 - 2\kappa_1 \partial_\eta^{-1} w_{2\xi} + 2\kappa_2 \phi_\eta \partial_\xi^{-1} w_1 + v_0. \tag{3.10}$$

The condition of equality of mixture derivatives $v_{0\xi\eta}$ and $v_{0\eta\xi}$, calculated from (3.4) and (3.5), leads to the following nonlinear evolution equation in terms of gauge invariants w_1 and w_2,

$$w_{1t} = -\kappa_1 w_{1\xi\xi} + \kappa_2 w_{1\eta\eta} - 2\kappa_1 w_{2\xi\xi} + 2\kappa_2 w_{2\eta\eta} - 2\kappa_1 \left(w_1 \partial_\eta^{-1} w_1\right)_\xi + 2\kappa_2 \left(w_1 \partial_\xi^{-1} w_1\right)_\eta. \tag{3.11}$$

One can show also that the exclusion of free field variable v_0 from the last equation (3.6) by the use of derivatives $v_{0\xi}$, $v_{0\eta}$ and $v_{0\xi\eta}$, calculated from the first two equations (3.4) and (3.5), leads to another evolution equation in terms of invariants w_1 and w_2,

$$w_{2t} = \kappa_1 w_{2\xi\xi} - \kappa_2 w_{2\eta\eta} - 2\kappa_1\big(w_2\partial_\eta^{-1}w_1\big)_\xi + 2\kappa_2\big(w_2\partial_\xi^{-1}w_1\big)_\eta. \tag{3.12}$$

So by the change of variables (3.7), (3.8) the integrable system (3.4)–(3.6) of nonlinear equations of second order is reduced to the following equivalent integrable system of nonlinear equations:

$$\rho_t = -\kappa_1\rho_{\xi\xi} - \kappa_2\rho_{\eta\eta} - 2\kappa_1\rho\partial_\eta^{-1}w_{2\xi} + 2\kappa_2\rho_\eta\partial_\xi^{-1}w_1 + v_0\rho, \tag{3.13}$$

$$w_{1t} = -\kappa_1 w_{1\xi\xi} + \kappa_2 w_{1\eta\eta} - 2\kappa_1 w_{2\xi\xi} + 2\kappa_2 w_{2\eta\eta} - 2\kappa_1\big(w_1\partial_\eta^{-1}w_1\big)_\xi + 2\kappa_2\big(w_1\partial_\xi^{-1}w_1\big)_\eta, \tag{3.14}$$

$$w_{2t} = \kappa_1 w_{2\xi\xi} - \kappa_2 w_{2\eta\eta} - 2\kappa_1\big(w_2\partial_\eta^{-1}w_1\big)_\xi + 2\kappa_2\big(w_2\partial_\xi^{-1}w_1\big)_\eta. \tag{3.15}$$

In terms of variables $\phi = \ln\rho$, w_1 and w_2 the integrable system (3.13)–(3.15) takes the form

$$\phi_t = -\kappa_1\phi_{\xi\xi} - \kappa_2\phi_{\eta\eta} - \kappa_1(\phi_\xi)^2 - \kappa_2(\phi_\eta)^2 - 2\kappa_1\partial_\eta^{-1}w_{2\xi} + 2\kappa_2\phi_\eta\partial_\xi^{-1}w_1 + v_0, \tag{3.16}$$

$$w_{1t} = -\kappa_1 w_{1\xi\xi} + \kappa_2 w_{1\eta\eta} - 2\kappa_1 w_{2\xi\xi} + 2\kappa_2 w_{2\eta\eta} - 2\kappa_1\big(w_1\partial_\eta^{-1}w_1\big)_\xi + 2\kappa_2\big(w_1\partial_\xi^{-1}w_1\big)_\eta, \tag{3.17}$$

$$w_{2t} = \kappa_1 w_{2\xi\xi} - \kappa_2 w_{2\eta\eta} - 2\kappa_1\big(w_2\partial_\eta^{-1}w_1\big)_\xi + 2\kappa_2\big(w_2\partial_\xi^{-1}w_1\big)_\eta. \tag{3.18}$$

In terms of variables $\phi = \ln\rho$, w_2 and $\widetilde{w}_2 = w_2 + w_1$ the integrable system (3.13)–(3.15) converts into more symmetrical form

$$\phi_t = -\kappa_1\phi_{\xi\xi} - \kappa_2\phi_{\eta\eta} - \kappa_1(\phi_\xi)^2 - \kappa_2(\phi_\eta)^2 - 2\kappa_1\partial_\eta^{-1}w_{2\xi} + 2\kappa_2\phi_\eta\partial_\xi^{-1}w_1 + v_0, \tag{3.19}$$

$$w_{2t} = \kappa_1 w_{2\xi\xi} - \kappa_2 w_{2\eta\eta} - 2\kappa_1\big(w_2\partial_\eta^{-1}(\widetilde{w}_2 - w_2)\big)_\xi + 2\kappa_2\big(w_2\partial_\xi^{-1}(\widetilde{w}_2 - w_2)\big)_\eta, \tag{3.20}$$

$$\widetilde{w}_{2t} = -\kappa_1\widetilde{w}_{2\xi\xi} + \kappa_2\widetilde{w}_{2\eta\eta} - 2\kappa_1\big(\widetilde{w}_2\partial_\eta^{-1}(\widetilde{w}_2 - w_2)\big)_\xi + 2\kappa_2\big(\widetilde{w}_2\partial_\xi^{-1}(\widetilde{w}_2 - w_2)\big)_\eta. \tag{3.21}$$

Remember for convenience that due to (1.5)–(1.8) in equivalent to each other systems of nonlinear equations (3.13)–(3.15), (3.16)–(3.18) and (3.19)–(3.21) the variables $\phi = \ln\rho, w_1, w_2$ and $\widetilde{w}_2$ are connected with the field variables u_1, v_1, u_0 of the linear problem (1.1) by the formulae

$$u_1 = \frac{\rho_\eta}{\rho} = \phi_\eta, \qquad v_1 = \frac{\rho_\xi}{\rho} - \partial_\eta^{-1}w_1 = \phi_\xi - \partial_\eta^{-1}w_1, \qquad w_1 = u_{1\xi} - v_{1\eta}, \tag{3.22}$$

$$w_2 = u_0 - u_{1\xi} - u_1v_1 = u_0 - \phi_{\xi\eta} - \phi_\eta\phi_\xi + \phi_\eta\partial_\eta^{-1}w_1 = u_0 - \frac{\rho_{\xi\eta}}{\rho} + \frac{\rho_\eta}{\rho}\partial_\eta^{-1}w_1, \tag{3.23}$$

$$\widetilde{w}_2 = w_2 + w_1. \tag{3.24}$$

Integrable system of nonlinear equations (3.13)–(3.15) (and analogously equivalent systems (3.16)–(3.18) or (3.19)–(3.21)) for the choice of variables

$$\rho = 1; \qquad u_1 = 0, v_1 = v, u_0 = u; \qquad v_0 = 2\kappa_1\partial_\eta^{-1}w_{2\xi} \tag{3.25}$$

for which $w_1 = -v_\eta$, $w_2 = u$, reduces to known system of equations

$$v_t = -\kappa_1 v_{\xi\xi} + \kappa_2 v_{\eta\eta} - 2\kappa_2 u_\eta + 2\kappa_1 v v_\xi + 2\kappa_1\partial_\eta^{-1}u_{\xi\xi} - 2\kappa_2 v_\eta\partial_\xi^{-1}v_\eta, \tag{3.26}$$

$$u_t = \kappa_1 u_{\xi\xi} - \kappa_2 u_{\eta\eta} + 2\kappa_1 (uv)_\xi - 2\kappa_2 \left(u\partial_\xi^{-1} v_\eta\right)_\eta, \tag{3.27}$$

derived in different context by Konopelchenko [22].

For the particular values $\kappa_1 = 1$ and $\kappa_2 = 0$, system of equations (3.26)–(3.27) reduces to famous integrable two-dimensional generalization of dispersive long-wave system of equations

$$v_{t\eta} = -v_{\xi\xi\eta} + 2u_{\xi\xi} + (v^2)_{\xi\eta}, \tag{3.28}$$

$$u_t = u_{\xi\xi} + 2(uv)_\xi, \tag{3.29}$$

discovered by Boiti, Leon and Pempinelli [18]. It is interesting to note that in a different context the system of equations (3.20)–(3.21) for Laplace invariants $h = w_2$ and $k = \widetilde{w}_2$ in the case $\kappa_1 = 1, \kappa_2 = 0$ in the paper of Weiss [25] was considered. By this reason and due to the remarks in section 1 (see (1.13)–(1.22) and discussion therein) it is worthwhile to name the integrable system of nonlinear equations (3.13)–(3.15) (and analogously equivalent systems (3.16)–(3.18) or (3.19)–(3.21)) as a two-dimensional generalization of dispersive long-wave (2DGDLW) system of equations.

All considered equivalent to each other, 2DGDLW integrable systems of nonlinear equations (3.13)–(3.15), (3.16)–(3.18) and (3.19)–(3.21) have a common gauge-transparent structure:

- they contain explicitly gauge-invariant subsystems (3.14)–(3.15), (3.17)–(3.18) of nonlinear equations for gauge invariants w_1 and w_2 (or equivalently subsystem (3.20)–(3.21) for gauge invariants w_2 and $\widetilde{w}_2$);
- they include equation (3.13) for pure gauge variable ρ (or equation (3.16) for variable $\phi = \ln \rho$) (with simple rule of gauge transformation $\rho \to \rho' = g\rho$) with additional terms containing gauge invariants and field variable v_0.

Such structure of 2DGDLW systems reflects existing gauge freedom in auxiliary linear problems (1.1) and (1.2).

Due to formulae (1.5), (1.7) and (3.1)–(3.3) 2DGDLW system (3.13)–(3.15) has triad representation $[L_1, L_2] = B(w_1)L_1$ with operators L_1, L_2 and coefficient $B(w_1)$ of the following forms:

$$L_1 = \partial_{\xi\eta}^2 + \frac{\rho_\eta}{\rho}\partial_\xi + \left(\frac{\rho_\xi}{\rho} - \left(\partial_\eta^{-1} w_1\right)\right)\partial_\eta + w_2 + \frac{\rho_{\xi\eta}}{\rho} - \frac{\rho_\eta}{\rho}\partial_\eta^{-1} w_1, \tag{3.30}$$

$$L_2 = \partial_t + \kappa_1 \partial_\xi^2 + \kappa_2 \partial_\eta^2 + 2\kappa_1 \frac{\rho_\xi}{\rho}\partial_\xi + 2\kappa_2 \left(\frac{\rho_\eta}{\rho} - \left(\partial_\xi^{-1} w_1\right)\right)\partial_\eta + v_0, \tag{3.31}$$

$$B(w_1) = 2\kappa_1 \partial_\eta^{-1} w_{1\xi} - 2\kappa_2 \partial_\xi^{-1} w_{1\eta}. \tag{3.32}$$

Let us consider some particular gauges of established 2DGDLW systems of equations (3.13)–(3.15), (3.16)–(3.18) and (3.19)–(3.21). It is convenient to denote the gauge in general position by the symbol $C\,(u_1, v_1, u_0)$.

In the gauge $C\left(u_1 = \phi_\eta, v_1 = \phi_\xi, u_0 = \phi_{\xi\eta} + \phi_\xi\phi_\eta\right)$ which due to (1.5)–(1.7) corresponds to zero values of invariants w_1 and w_2

$$w_1 = u_{1\xi} - v_{1\eta} = 0, \qquad w_2 = u_0 - u_{1\xi} - u_1 v_1 = 0, \qquad \widetilde{w}_2 = 0, \tag{3.33}$$

the 2DGDLW system of equations (3.19)–(3.21) reduces to two-dimensional Burgers equation in potential form

$$\phi_t = -\kappa_1 \phi_{\xi\xi} - \kappa_2 \phi_{\eta\eta} - \kappa_1 (\phi_\xi)^2 - \kappa_2 (\phi_\eta)^2 + v_0, \tag{3.34}$$

or in terms of variable ρ connected with ϕ by Hopf–Cole transformation $\phi = \ln\rho$, to linear diffusion equation

$$\rho_t = -\kappa_1\rho_{\xi\xi} - \kappa_2\rho_{\eta\eta} + v_0\rho. \tag{3.35}$$

Equation (3.34) (or (3.35)) due to our construction is a compatibility condition in Lax form

$$[L_1, L_2] = B(w_1)L_1 \equiv 0 \tag{3.36}$$

of linear problems (1.1) and (1.2) with operators L_1, L_2 given by (3.30), (3.31) under substitution $w_1 = w_2 = 0$.

In another simple gauge $C\left(u_1 = \phi_\eta, v_1 = 0, u_0 = 0\right)$ corresponding due to (3.22)–(3.24) to the invariants

$$w_1 = \phi_{\xi\eta}, \qquad w_2 = -\phi_{\xi\eta}, \qquad \widetilde{w}_2 = 0, \tag{3.37}$$

the 2DGDLW system of equations (3.19)–(3.21) for the choice $v_0 = 0$ again reduces to the single equation of Burgers type in potential form

$$\phi_t = \kappa_1\phi_{\xi\xi} - \kappa_2\phi_{\eta\eta} - \kappa_1(\phi_\xi)^2 + \kappa_2(\phi_\eta)^2. \tag{3.38}$$

This equation linearizes by Hopf–Cole transformation $\phi = -\ln\rho$ to the corresponding linear equation

$$\rho_t = \kappa_1\rho_{\xi\xi} - \kappa_2\rho_{\eta\eta}. \tag{3.39}$$

In the less trivial gauge $C(u_1 = 0, v_1 = -q_\xi/q, u_0 = pq)$ the invariants w_1, w_2 and $\widetilde{w}_2$ due to (3.22)–(3.24) are given by the following expressions:

$$w_1 = (\ln q)_{\xi\eta}, \qquad w_2 = u_0 = pq, \qquad \widetilde{w}_2 = pq + (\ln q)_{\xi\eta}, \tag{3.40}$$

the variable ρ due to (3.22) has constant value, consequently the variable $\phi = 0$. In this case due to (3.19)

$$v_0 = 2\kappa_1\partial_\eta^{-1}w_{2\xi} = 2\kappa_1\partial_\eta^{-1}(pq)_\xi \tag{3.41}$$

and from the 2DGDLW system of equations (3.19)–(3.21) one obtains after some calculations the famous DS system of equations [19] for the field variables p and q,

$$p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} + 2\kappa_1 p\partial_\eta^{-1}(pq)_\xi - 2\kappa_2 p\partial_\xi^{-1}(pq)_\eta, \tag{3.42}$$

$$q_t = -\kappa_1 q_{\xi\xi} + \kappa_2 q_{\eta\eta} - 2\kappa_1 q\partial_\eta^{-1}(pq)_\xi + 2\kappa_2 q\partial_\xi^{-1}(pq)_\eta. \tag{3.43}$$

One can consider also the gauge $C(u_1 = p_\eta, v_1 = q_\xi, u_0 = p_\eta q_\xi)$ in which due to (3.22)–(3.24) the invariants have the following expressions through q and p:

$$w_1 = p_{\xi\eta} - q_{\xi\eta}, \qquad w_2 = -p_{\xi\eta}, \qquad \widetilde{w}_2 = -q_{\xi\eta}. \tag{3.44}$$

Substitution of w_1, w_2 and $\widetilde{w}_2$ from (3.44) into the system (3.19)–(3.21) leads to the following three equations for p and q. From equation (3.19) for $\phi \equiv p$ one obtains

$$p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} - \kappa_1(p_\xi)^2 + \kappa_2(p_\eta)^2 - 2\kappa_2 p_\eta q_\eta + v_0. \tag{3.45}$$

Equations (3.20) and (3.21) for w_2 and $\widetilde{w}_2$ in terms of variables p, q take the forms

$$p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} - \kappa_1(p_\xi)^2 + \kappa_2(p_\eta)^2 + 2\kappa_1\partial_\eta^{-1}(p_{\xi\eta}q_\xi) - 2\kappa_2\partial_\xi^{-1}(p_{\xi\eta}q_\eta), \tag{3.46}$$

$$q_t = -\kappa_1 q_{\xi\xi} + \kappa_2 q_{\eta\eta} + \kappa_1(q_\xi)^2 - \kappa_2(q_\eta)^2 - 2\kappa_1\partial_\eta^{-1}(q_{\xi\eta}p_\xi) + 2\kappa_2\partial_\xi^{-1}(q_{\xi\eta}p_\eta). \tag{3.47}$$

Equations (3.45) and (3.46) are compatible for the choice of v_0 given by the formula

$$v_0 = 2\kappa_1\partial_\eta^{-1}(p_{\xi\eta}q_\xi) + 2\kappa_2\partial_\xi^{-1}(q_{\xi\eta}p_\eta), \tag{3.48}$$

and the system of three equations (3.45)–(3.47) reduces to the system of two equations (3.46)–(3.47) containing in nonlocal terms derivatives $p_{\xi\eta}q_\xi$, $p_{\xi\eta}q_\eta$, etc.

Analogously in the gauge $C(u_1 = p_\eta, v_1 = q_\xi, u_0 = 0)$ it follows for w_1, w_2 and $\widetilde{w}_2$ due to (3.22)–(3.24)

$$w_1 = p_{\xi\eta} - q_{\xi\eta}, \qquad w_2 = -p_{\xi\eta} - p_\eta q_\xi, \qquad \widetilde{w}_2 = -q_{\xi\eta} - p_\eta q_\xi. \tag{3.49}$$

Equation (3.16) for $\phi \equiv p$ via (3.49) takes the form

$$p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} - \kappa_1 (p_\xi)^2 + \kappa_2 (p_\eta)^2 - 2\kappa_2 p_\eta q_\eta + 2\kappa_1 \partial_\eta^{-1}(p_\eta q_\xi)_\xi + v_0. \tag{3.50}$$

Equation (3.17) via substitutions from (3.49) transforms to the form

$$\begin{aligned} p_t - q_t = {}& \kappa_1 (p+q)_{\xi\xi} - \kappa_2 (p+q)_{\eta\eta} - \kappa_1 (p_\xi - q_\xi)^2 + \kappa_2 (p_\eta - q_\eta)^2 \\ & + 2\kappa_1 \partial_\eta^{-1}(p_\eta q_\xi)_\xi - 2\kappa_2 \partial_\xi^{-1}(p_\eta q_\xi)_\eta. \end{aligned} \tag{3.51}$$

By substraction of equation (3.51) from equation (3.50) one obtains the evolution equation for q:

$$q_t = -\kappa_1 q_{\xi\xi} + \kappa_2 q_{\eta\eta} + \kappa_1 (q_\xi)^2 - \kappa_2 (q_\eta)^2 - 2\kappa_1 p_\xi q_\xi + 2\kappa_2 \partial_\xi^{-1}(p_\eta q_\xi)_\eta + v_0. \tag{3.52}$$

Equation (3.18) for the invariant w_2 due to (3.49) in terms of variables p, q is

$$\begin{aligned} (p_{\xi\eta} + p_\eta q_\xi)_t = {}& \kappa_1 (p_{\xi\eta} + p_\eta q_\xi)_{\xi\xi} - \kappa_2 (p_{\xi\eta} + p_\eta q_\xi)_{\eta\eta} \\ & - 2\kappa_1 ((p_{\xi\eta} + p_\eta q_\xi)(p_\xi - q_\xi))_\xi + 2\kappa_2 ((p_{\xi\eta} + p_\eta q_\xi)(p_\eta - q_\eta))_\eta. \end{aligned} \tag{3.53}$$

Equations (3.50), (3.52) and (3.53) are compatible with each other if the field variable v_0 satisfies the equation

$$v_{0\xi\eta} + p_\eta v_{0\xi} + q_\xi v_{0\eta} = 0. \tag{3.54}$$

For the simple choice $v_0 \equiv 0$ one obtains from the system of the three equations (3.50), (3.52) and (3.53) the following equivalent system of two equations:

$$p_t = \kappa_1 p_{\xi\xi} - \kappa_2 p_{\eta\eta} - \kappa_1 (p_\xi)^2 + \kappa_2 (p_\eta)^2 - 2\kappa_2 p_\eta q_\eta + 2\kappa_1 \partial_\eta^{-1}(p_\eta q_\xi)_\xi, \tag{3.55}$$

$$q_t = -\kappa_1 q_{\xi\xi} + \kappa_2 q_{\eta\eta} + \kappa_1 (q_\xi)^2 - \kappa_2 (q_\eta)^2 - 2\kappa_1 p_\xi q_\xi + 2\kappa_2 \partial_\xi^{-1}(p_\eta q_\xi)_\eta. \tag{3.56}$$

At first this system of equations has been derived in another context in the paper of Konopelchenko [22].

In conclusion, let us derive Miura-type transformations between different systems of DS-type equations of second order obtained in this section in different gauges. For convenience let us denote by capital letters $P \equiv p$, $Q \equiv q$ the solutions of the DS famous system (3.43)–(3.43) of equations. By the use of invariants w_1 and w_2 one obtains the following relations between variables $(P \equiv p, Q \equiv q)$ of DS system (3.42)–(3.43) and variables p, q of the system (3.46)–(3.47),

$$w_1 = (\ln Q)_{\xi\eta} = p_{\xi\eta} - q_{\xi\eta}, \qquad w_2 = PQ = -p_{\xi\eta}. \tag{3.57}$$

One derives from (3.57),

$$Q = \mathrm{e}^{p-q}, \qquad P = -p_{\xi\eta}\,\mathrm{e}^{q-p}. \tag{3.58}$$

Quite analogously for the pair of DS systems (3.42)–(3.43) and (3.55)–(3.56) one has

$$w_1 = (\ln Q)_{\xi\eta} = p_{\xi\eta} - q_{\xi\eta}, \qquad w_2 = PQ = -p_{\xi\eta} - p_\eta q_\xi. \tag{3.59}$$

One obtains from (3.59),

$$Q = \mathrm{e}^{p-q}, \qquad P = -(p_{\xi\eta} + p_\eta q_\xi)\,\mathrm{e}^{q-p}. \tag{3.60}$$

Transformations (3.58) and (3.60) allow us to obtain solutions of the famous DS system of equations (3.42)–(3.43) from the systems of equations (3.46)–(3.47) and (3.55)–(3.56), these transformations are Miura-type transformations being gauge-equivalent to other DS-type systems of equations of second order.

4. Conclusion

In conclusion let us underline once again that ideas of gauge invariance now are in common use in the theory of integrable nonlinear evolution equations. There are known attempts to develop invariant description of some nonlinear integrable equations considered in the present paper by the use of matrix linear auxiliary problems. This was done for example in the paper [26] for the Nizhnik–Veselov–Novikov and Davey–Stewartson equations in the framework of the classical invariant theory of second-order linear partial differential equations.

Matrix linear auxiliary problems have a bigger number of degrees of freedom than the scalar, the performance of reductions from general position to integrable nonlinear equations is more difficult; all this leads to the need of consideration gauge transformations under some restrictions, manifestly the gauge-invariant description of integrable nonlinear equations in this case is far from completion and requires additional research work.

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